APPENDIX B

CLEBSCH–GORDAN COEFFICIENTS

Clebsch–Gordan coefficients arise when two angular momenta are combined into a total angular momentum. This will occur when the angular momentum of a system is found as the combination of the angular momenta of two subsystems or when two types of angular momenta relating to the same particle are combined to find the total angular momentum for that particle, as in the addition of orbital and spin angular momenta to obtain a total angular momentum for the particle.

Let us find the total angular momentum $\mathbf{j}$ as a sum of the momenta $\mathbf{j}_1$ and $\mathbf{j}_2$. The wave function for the total momentum can be written as

$$\Psi_{jm} = \sum_{m_1, m_2} \langle j_1 j_2, m_1 m_2 \,|jm\rangle \, \psi_{j_1 m_1} \psi_{j_2 m_2}, \tag{B.1}$$

where the indices on the wave functions characterize the angular momentum and its projection onto a fixed axis. The coefficients in this expansion are called Clebsch–Gordan coefficients. We now examine their properties.

B.1 PROPERTIES OF CLEBSCH–GORDAN COEFFICIENTS

B.1.1 Condition for Addition of Angular Momentum Projections

It follows from the conservation law for the sum of angular momentum projections that

$$\langle j_1 j_2, m_1 m_2 \,|jm\rangle = 0 \qquad \text{if} \qquad m_1 + m_2 \neq m. \tag{B.2}$$

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B.1.2 Orthogonality Condition

Orthogonality properties of the wave functions are expressed as

$$\langle \Psi_{jm} | \Psi_{jm'} \rangle = \delta_{mm'},$$
$$\langle \psi_{j_1 m_1} | \psi_{j_1 m_1'} \rangle = \delta_{m_1 m_1'},$$

This leads to the orthogonality condition for the Clebsch–Gordan coefficients, which is

$$\sum_{m_1 m_2} \langle j_1 j_2, m_1 m_2 \,|\, jm \rangle \langle j_1 j_2, m_1 m_2 \,|\, jm' \rangle = \delta_{mm'}, \tag{B.3}$$

where $\delta_{mm'}$ is the Kronecker delta symbol defined by

$$\delta_{mm'} = \begin{cases} 1, & m = m' \\ 0, & m \neq m' \end{cases}.$$

B.1.3 Inversion Property

In the inversion transformation of the radius vector, $\mathbf{r} \to -\mathbf{r}$, the wave functions will change sign or not, depending on their parity. That is, the wave functions transform as

$$\Psi_{jm} \to (-1)^{j-m} \Psi_{j,-m}.$$

This leads to

$$\begin{aligned} (-1)^{j-m} \Psi_{j,-m} &= (-1)^{j-m} \sum_{m_1 m_2} \langle j_1 j_2, -m_1, -m_2 \,|\, j, -m \rangle \psi_{j_1,-m_1} \psi_{j_2,-m_2} \\ &= \sum_{m_1 m_2} \langle j_1 j_2, m_1 m_2 \,|\, jm \rangle \psi_{j_1 m_1} \psi_{j_2 m_2}, \end{aligned}$$

so that we obtain

$$\langle j_1 j_2, -m_1, -m_2 \,|\, j, -m \rangle = (-1)^{j-j_1-j_2} \langle j_1 j_2, m_1 m_2 \,|\, jm \rangle. \tag{B.4}$$

B.1.4 Permutation Properties

It follows from the rules for the construction of the Clebsch–Gordan coefficients and the rules for the addition of two angular momenta into a zero total angular momentum that

$$\begin{aligned} \langle j_1 j_2, m_1 m_2 \,|\, jm \rangle &= (-1)^{j_1-m_1} \sqrt{\frac{2j+1}{2j_2+1}} \langle j_1 j, m_1, -m \,|\, j_2 - m_2 \rangle \\ &= (-1)^{j_2+m_2} \sqrt{\frac{2j+1}{2j_1+1}} \langle j j_2, -m, m_2 \,|\, j_1 - m_1 \rangle. \end{aligned} \tag{B.5}$$

B.2 EVALUATION OF CERTAIN CLEBSCH–GORDAN COEFFICIENTS

We consider first the evaluation of Clebsch–Gordan coefficients for the frequently occurring case $j_2 = \frac{1}{2}$. We begin by obtaining a relation for Clebsch–Gordan coefficients that is valid for any angular momenta. From the addition properties used to form $j = j_1 + j_2$, we obtain the condition

$$\widehat{j}^2 - \widehat{j_1}^2 - \widehat{j_2}^2 = 2\widehat{\mathbf{j}}_1 \cdot \widehat{\mathbf{j}}_2 = 2\widehat{j_{1z}}\widehat{j_{2z}} + \widehat{j_{1+}}\widehat{j_{2-}} + \widehat{j_{1-}}\widehat{j_{2+}}.$$

When this operator acts on the wave function in Eq. (B.1), the result is

$$\begin{aligned} j(j+1) - j_1(j_1+1) - j_2(j_2+1) = \sum_{m_2 m_2'} &\langle j_1 j_2, m_1 m_2 | jm\rangle \langle j_1 j_2, m_1' m_2' | jm\rangle \\ &\times \left\langle \psi_{j_1 m_1'} \psi_{j_2 m_2'} | 2 j_{1z} j_{2z} + j_{1+} j_{2-} \right. \\ &\left. + j_{1-} j_{2+} | \psi_{j_1 m_1} \psi_{j_2 m_2} \right\rangle. \end{aligned}$$

For the given value of m, the conservation rule for angular momentum projections gives

$$m_1 = m - m_2, \qquad m_1' = m - m_2'.$$

With the help of the momentum operator eigenvalues in Eqs. (A.2) and (A.9), we find that

$$\begin{aligned} &j(j+1) - j_1(j_1+1) - j_2(j_2+1) \\ &\quad = 2\sum_{m_2} m_1 m_2 \langle j_1 j_2, m_1 m_2 | jm\rangle^2 \\ &\qquad + \sum_{m_2} \sqrt{(j_1 - m_1)(j_1 + m_1 + 1)(j_2 - m_2 + 1)(j_2 + m_2)} \\ &\qquad \times \langle j_1 j_2, m_1 m_2 | jm\rangle \langle j_1 j_2, m_1 + 1, m_2 - 1 | jm\rangle \\ &\qquad + \sum_{m_2} \sqrt{(j_1 - m_1 + 1)(j_1 + m_1)(j_2 - m_2)(j_2 + m_2 + 1)} \\ &\qquad \times \langle j_1 j_2, m_1 m_2 | jm\rangle \langle j_1 j_2, m_1 - 1, m_2 + 1 | jm\rangle. \end{aligned} \tag{B.6}$$

Now we apply the above results to the particular case where $j_2 = \frac{1}{2}$. For the given values j, j_1, there are two nonzero Clebsch–Gordan coefficients that we shall denote as

$$X = \left\langle j_1 \tfrac{1}{2}, m - \tfrac{1}{2}, \tfrac{1}{2} \, | jm \right\rangle,$$
$$Y = \left\langle j_1 \tfrac{1}{2}, m + \tfrac{1}{2}, -\tfrac{1}{2} \, | jm \right\rangle.$$

Using this notation, Eq. (B.6) for $j_2 = \frac{1}{2}$ becomes

$$j(j+1) - j_1(j_1+1) - \tfrac{3}{4} = \left(m - \tfrac{1}{2}\right)X^2 - \left(m + \tfrac{1}{2}\right)Y^2 + 2\sqrt{\left(j_1 + \tfrac{1}{2}\right)^2 - m^2}\,XY.$$

The normalization condition for the Clebsch–Gordan coefficients given in Eq. (B.3) is of the form

$$X^2 + Y^2 = 1.$$

When this is inserted into the preceding expression, we obtain

$$(j - j_1)(j + j_1 + 1) - \tfrac{1}{4} = m\left(X^2 - Y^2\right) + 2\sqrt{\left(j_1 + \tfrac{1}{2}\right)^2 - m^2}XY.$$

With the notation

$$t = \frac{m}{j + \frac{1}{2}} \leq 1,$$

we can rewrite our results as the system of equations

$$t(X^2 - Y^2) + 2\sqrt{1 - t^2}XY = \pm 1, \qquad X^2 + Y^2 = 1.$$

The ambiguous sign $\pm$ is such that the upper sign corresponds to $j = j_1 + \frac{1}{2}$, while the lower sign is associated with $j = j_1 - \frac{1}{2}$.

The solution of the equations we have obtained is

$$X = \sqrt{\frac{1+t}{2}}, \qquad Y = \sqrt{\frac{1-t}{2}}$$

for $j = j_1 + \frac{1}{2}$, and for $j = j_1 - \frac{1}{2}$ it is

$$X = \sqrt{\frac{1-t}{2}}, \qquad Y = -\sqrt{\frac{1+t}{2}}.$$

These results are summarized in Table B.1.

TABLE B.1. $\left\langle j_1 \tfrac{1}{2}, m - \sigma, \sigma \mid jm \right\rangle$

	σ	
$j - j_1$	$\frac{1}{2}$	$-\frac{1}{2}$
$\dfrac{1}{2}$	$\sqrt{\dfrac{j_1 + m + \frac{1}{2}}{2j_1 + 1}}$	$\sqrt{\dfrac{j_1 - m + \frac{1}{2}}{2j_1 + 1}}$
$-\dfrac{1}{2}$	$\sqrt{\dfrac{j_1 - m + \frac{1}{2}}{2j_1 + 1}}$	$-\sqrt{\dfrac{j_1 + m + \frac{1}{2}}{2j_1 + 1}}$

Other values for Clebsch–Gordan coefficients are generally more complicated to express. Thorough treatments of the relations between Clebsch–Gordan coefficients and tables of values are available in the literature. See, for example, Refs. 9 or 10. Another relatively simple form taken by the Clebsch–Gordan coefficients occurs when the projection of the angular momentum coincides with this momentum. In that case we have

$$\langle j_1 j_2, j_1 m_2 \,|\, jm\rangle = \frac{\delta_{j_1+m_2,m}}{\sqrt{(j_1+j_2+j+1)!}} \times \sqrt{\frac{(j_2-m_2)!\,(j+m)!\,(2j+1)\,(2j_1)!\,(j-j_1+j_2)!}{(j_2+m_2)!\,(j-m)!\,(j_1-j_2+j)!\,(j_1+j_2-j)!}}, \tag{B.7}$$

$$\langle j_1 j_2, m_1 j_2 \,|\, jm\rangle = \frac{(-1)^{j_1+j_2-j}\,\delta_{m_1+j_2,m}}{\sqrt{(j_1+j_2+j+1)!}} \times \sqrt{\frac{(j_1-m_1)!\,(j+m)!\,(2j+1)\,(2j_2)!\,(j_1-j_2+j)!}{(j_1+m_1)!\,(j-m)!\,(j-j_1+j_2)!\,(j_1+j_2-j)!}}, \tag{B.8}$$

$$\langle j_1 j_2, -j_1 m_2 \,|\, jm\rangle = (-1)^{j_1+j_2-j}\langle j_1 j_2, j_1 - m_2 \,|\, j-m\rangle, \tag{B.9}$$

$$\langle j_1 j_2, m_1 - j_2 \,|\, jm\rangle = (-1)^{j_1+j_2-j+1}\langle j_1 j_2, -m_1 j_2 \,|\, j-m\rangle. \tag{B.10}$$

Values of the Clebsch–Gordan coefficients are given in Table B.2 for the case $j_2 = 1$. If $m_2 = 1$ or -1, then the coefficients can be calculated using the connections in Eqs. (B.9) and (B.10). The third possibility, $m_2 = 0$, can be calculated using the values obtained for $m_2 = 1$ and -1, and the normalization condition (B.3) for the Clebsch–Gordan coefficients.

TABLE B.2. $\langle j_1 1; m - m_2, m_2 \mid jm\rangle$

	m_2		
$j - j_1$	1	0	-1
1	$\sqrt{\frac{(j_1+m+1)(j_1+m)}{(2j_1+2)(2j_1+1)}}$	$\sqrt{\frac{(j_1-m+1)(j_1+m+1)}{(2j_1+1)(j_1+1)}}$	$\sqrt{\frac{(j_1-m+1)(j_1-m)}{(2j_1+2)(2j_1+1)}}$
0	$-\sqrt{\frac{(j_1-m+1)(j_1+m)}{2j_1(2j_1+1)}}$	$\frac{m}{\sqrt{j_1(j_1+1)}}$	$\sqrt{\frac{(j_1+m+1)(j_1-m)}{2j_1(j_1+1)}}$
-1	$\sqrt{\frac{(j_1-m+1)(j_1-m)}{2j_1(2j_1+1)}}$	$-\sqrt{\frac{(j_1-m)(j_1+m)}{j_1(2j_1+1)}}$	$\sqrt{\frac{(j_1+m+1)(j_1+m)}{2j_1(2j_1+1)}}$

B.3 WIGNER $3j$ SYMBOLS

A quantity closely related to the Clebsch–Gordan coefficient is the Wigner $3j$ coefficient, designed to achieve maximum symmetry. It can be defined as

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1 - j_2 - m_3} \langle j_1 j_2, m_1 m_2 \,|j_3, -m_3 \rangle . \tag{B.11}$$

The $3j$ symbol has the property that

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = 0 \qquad \text{if} \qquad m_1 + m_2 + m_3 \neq 0,$$

in place of Eq. (B.2).

We list the principal symmetry and orthogonality properties. Even permutation of the columns leaves the $3j$ symbol unchanged, or

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_2 & j_3 & j_1 \\ m_2 & m_3 & m_1 \end{pmatrix} = \begin{pmatrix} j_3 & j_1 & j_2 \\ m_3 & m_1 & m_2 \end{pmatrix} .$$

Odd permutation of the columns, on the other hand, is equivalent to multiplication by $(-1)^{j_1 + j_2 + j_3}$, so that

$$\begin{aligned} (-1)^{j_1 + j_2 + j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} &= \begin{pmatrix} j_2 & j_1 & j_3 \\ m_2 & m_1 & m_3 \end{pmatrix} \\ &= \begin{pmatrix} j_1 & j_3 & j_2 \\ m_1 & m_3 & m_2 \end{pmatrix} = \begin{pmatrix} j_3 & j_2 & j_1 \\ m_3 & m_2 & m_1 \end{pmatrix} . \end{aligned}$$

Orthogonality properties of the Wigner $3j$ symbols are

$$\begin{aligned} \sum_{j_3 m_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1' & m_2' & m_3 \end{pmatrix} &= \delta_{m_1 m_1'} \delta_{m_2 m_2'}, \\ \sum_{m_1 m_2} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3' \\ m_1 & m_2 & m_3' \end{pmatrix} &= \frac{\delta_{j_3 j_3'} \delta_{m_3 m_3'} \delta\,(j_1 j_2 j_3)}{2 j_3 + 1}, \end{aligned} \tag{B.12}$$

where $\delta\,(j_1 j_2 j_3)$ in Eq. (B.12) is a quantity defined as

$$\delta\,(j_1 j_2 j_3) = \begin{cases} 1 & \text{if } |j_1 - j_2| \leq j_3 \leq j_1 + j_2 \\ 0 & \text{otherwise} \end{cases} . \tag{B.13}$$

The statement in Eq. (B.13) is called the *triangular condition*.